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Recurrence Coefficients of a New Generalization of the Meixner Polynomials*

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Abstract. We investigate new generalizations of the Meixner polynomials on the lattice $\mathbb{N}$, on the shifted lattice $\mathbb{N}+1-\beta$ and on the bi-lattice $\mathbb{N}\cup(\mathbb{N}+1-\beta)$. We show that the coefficients of the three-term recurrence relation for the orthogonal polynomials are related to the solutions of the fifth Painlevé equation $\mathrm{P_V}$. Initial conditions for different lattices can be transformed to the classical solutions of $\mathrm{P_V}$ with special values of the parameters. We also study one property of the Bäcklund transformation of $\mathrm{P_V}$.

Key words: Painlevé equations; Bäcklund transformations; classical solutions; orthogonal polynomials; recurrence coefficients

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1 Introduction

The recurrence coefficients of orthogonal polynomials for semi-classical weights are often related to the Painlevé type equations (e.g., [10, 4, 5] and see also the overview in [2]). In this paper we study a new generalization of the Meixner weight. The recurrence coefficients of the corresponding orthogonal polynomials can be viewed as functions of one of the parameters. We show that the recurrence coefficients are related to solutions of the fifth Painlevé equation. Another generalization of the Meixner weight is presented in [2].

The paper is organized as follows. In the introduction we shall first review orthogonal polynomials for the generalized Meixner weight on different lattices and their main properties following [12]. Next we shall briefly recall the fifth Painlevé equation and its Bäcklund transformation. Further, by using the Toda system, we show that the recurrence coefficients can be expressed in terms of solutions of the fifth Painlevé equation. Finally we study initial conditions of the recurrence coefficients for different lattices and describe one property of the Bäcklund transformation of $\mathrm{P_V}$.

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1.1 Orthogonal polynomials for the generalized Meixner weight

One of the most important properties of orthogonal polynomials is the three-term recurrence relation. Let us consider a sequence $(p_n)_{n\in\mathbb{N}}$ of orthonormal polynomials for the weight w on the lattice $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$

$$\sum_{k=0}^{\infty} p_n(k)p_k(k)w(k) = \delta_{n,k},$$

where $\delta_{n,k}$ is the Kronecker delta. This relation takes the following form:

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x). \tag{1}$$

For the monic polynomials P_n related to orthonormal polynomials $p_n(x) = \gamma_n x^n + \cdots$ with

$$\frac{1}{\gamma_n^2} = \sum_{k=0}^{\infty} P_n^2(k)w(k),$$

the recurrence relation is given by

$$xP_n(x) = P_{n+1}(x) + b_n P_n(x) + a_n^2 P_{n-1}(x).$$

The classical Meixner polynomials ([3, Chapter VI], [12]) are orthogonal on the lattice $\mathbb{N}$ with respect to the negative binomial (or Pascal) distribution:

$$\sum_{k=0}^{\infty} M_n(k;\beta,c)M_m(k;\beta,c)\frac{(\beta)_k c^k}{k!} = \frac{c^{-n}n!}{(\beta)_n(1-c)^\beta}\,\delta_{n,m}, \qquad \beta > 0, \quad 0 < c < 1.$$

Here the Pochhammer symbol is defined by

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = x(x+1)\cdots(x+n-1).$$

The weight $w_k = w(k) = (\beta)_k c^k/k!$ satisfies the Pearson equation

$$\nabla[(\beta+x)w(x)] = \left(\beta + x - \frac{x}{c}\right) w(x),$$

where ∇ is the backward difference operator

$$\nabla f(x) = f(x) - f(x-1).$$

Here the function $w(x) = \Gamma(\beta+x)c^x/(\Gamma(\beta)\Gamma(x+1))$ gives the weights $w_k = w(k)$. The Pearson equation for the Meixner polynomials is, hence, of the form

$$\nabla[\sigma(x)w(x)] = \tau(x)w(x) \tag{2}$$

with $\sigma(x) = \beta + x$ and τ is a polynomial of degree 1.

Recall that the classical orthogonal polynomials are characterized by the Pearson equation (2) with σ a polynomial of degree at most 2 and τ a polynomial of degree 1. Note that in (2) the operator ∇ is used for orthogonal polynomials on the lattice and it is replaced by differentiation in case of orthogonal polynomials on an interval of the real line. The Pearson equation plays an important role for classical orthogonal polynomials since it allows to find many useful properties

of these polynomials. It is known that the recurrence coefficients of the Meixner polynomials are given explicitly by

$$a_n^2 = \frac{n(n+\beta-1)c}{(1-c)^2}, \qquad b_n = \frac{n+(n+\beta)c}{1-c}, \qquad n \in \mathbb{N}.$$

The Meixner weight can be generalized [12]. One can use the weight function

$$w(x) = \frac{\Gamma(\beta)\Gamma(\gamma+x)c^x}{\Gamma(\gamma)\Gamma(\beta+x)\Gamma(x+1)}$$

which gives the weight

$$w_k = w(k) = \frac{(\gamma)_k c^k}{(\beta)_k k!}, \qquad c, \beta, \gamma > 0, \tag{3}$$

on the lattice $\mathbb{N}$. The orthonormal polynomials $(p_n)_{n\in\mathbb{N}}$ for weight (3) satisfy

$$\sum_{k=0}^{\infty} p_n(k)p_m(k)w_k = 0, \qquad n \neq m. \tag{4}$$

The special case $\beta = 1$ was studied in [2]. The case $\beta = \gamma$ gives the well-known Charlier weight. The case $\gamma = 1$ corresponds to the classical Charlier weight on the shifted lattice $\mathbb{N} + 1 - \beta$.

Theorem 1 ([12, Theorem 3.1])**.** *The recurrence coefficients in the three-term recurrence relation* (1) *for the orthonormal polynomials defined by* (4)*, with respect to weight* (3) *on the lattice* $\mathbb{N}$*, satisfy*

$$a_n^2 = nc - (\gamma-1)u_n, \qquad b_n = n + \gamma - \beta + c - (\gamma-1)v_n/c,$$

where

$$(u_n+v_n)(u_{n+1}+v_n) = \frac{\gamma-1}{c^2}v_n(v_n-c)\left(v_n - c\frac{\gamma-\beta}{\gamma-1}\right), \tag{5}$$

$$(u_n+v_n)(u_n+v_{n-1}) = \frac{u_n}{u_n - cn/(\gamma-1)}(u_n+c)\left(u_n + c\frac{\gamma-\beta}{\gamma-1}\right), \tag{6}$$

with initial conditions

$$a_0^2 = 0, \qquad b_0 = \frac{\gamma c}{\beta}\frac{M(\gamma+1,\beta+1,c)}{M(\gamma,\beta,c)}, \tag{7}$$

where $M(a,b,z)$ *is the confluent hypergeometric function* ${}_1F_1(a;b;z)$*.*

The system (5), (6) can be identified as a limiting case of an asymmetric discrete Painlevé equation [12]. In this paper we show that it can be obtained from the Bäcklund transformation of the fifth Painlevé equation. Furthermore, one can use the weight (3) on the shifted lattice $\mathbb{N}+1-\beta$ and one can also combine both lattices to obtain the bi-lattice $\mathbb{N} \cup (\mathbb{N}+1-\beta)$. The orthogonality measure for the bi-lattice is a linear combination of the measures on $\mathbb{N}$ and $\mathbb{N}+1-\beta$.

The weight w in (3) on the shifted lattice $\mathbb{N}+1-\beta = \{1-\beta, 2-\beta, 3-\beta, \dots\}$ is, up to a constant factor, equal to the weight on the original lattice $\mathbb{N}$, with different parameters [12]. Denoting

$$w_{\gamma,\beta,c}(x) = \frac{\Gamma(\beta)}{\Gamma(\gamma)}\frac{\Gamma(\gamma+x)c^x}{\Gamma(x+1)\Gamma(\beta+x)}$$

one has

$$w_{\gamma,\beta,c}(k+1-\beta) = c^{1-\beta}\frac{\Gamma(\beta)\Gamma(\gamma+1-\beta)}{\Gamma(2-\beta)\Gamma(\gamma)}w_{\gamma+1-\beta,2-\beta,c}(k). \tag{8}$$

The corresponding orthonormal polynomials $(q_n)_{n\in\mathbb{N}}$ satisfy

$$\sum_{k=0}^{\infty} q_n(k+1-\beta)q_m(k+1-\beta)w(k+1-\beta) = 0, \qquad n \neq m.$$

Moreover, these polynomials are equal to the polynomials p_n shifted in both the variable x and the parameters β and γ. For the positivity of the weights $(w(k+1-\beta))_{k\in\mathbb{N}}$ it is necessary to have $c > 0$, $\beta < 2$, $\gamma > \beta - 1$. In [12, Theorem 3.2] it is shown that the recurrence coefficients in the three-term recurrence relation

$$xq_n(x) = \hat{a}_{n+1}q_{n+1}(x) + \hat{b}_n q_n(x) + \hat{a}_n q_{n-1}(x)$$

satisfy the same system (5), (6) (with hats) but with initial conditions

$$\hat{a}_0^2 = 0, \qquad \hat{b}_0 = (1-\beta)\frac{M(\gamma-\beta+1, 1-\beta, c)}{M(\gamma-\beta+1, 2-\beta, c)}. \tag{9}$$

Using the orthogonality measure $\mu = \mu_1 + \tau\mu_2$, where $\tau > 0$, μ_1 is the discrete measure on $\mathbb{N}$ with weights $w_k = w(k)$ and μ_2 is the discrete measure on $\mathbb{N}+1-\beta$ with weights $v_k = w(k+1-\beta)$, one can study orthonormal polynomials $(r_n)_{n\in\mathbb{N}}$, satisfying the three-term recurrence relation

$$xr_n(x) = \tilde{a}_{n+1}r_{n+1}(x) + \tilde{b}_n r_n(x) + \tilde{a}_n r_{n-1}(x).$$

One needs to impose the conditions $c > 0$, $0 < \beta < 2$, $\gamma > \max(0, \beta - 1)$ for the positivity of the measures. The orthogonality relation is given by

$$\sum_{k=0}^{\infty} r_n(k)r_m(k)w(k) + \tau\sum_{k=0}^{\infty} r_n(k+1-\beta)r_m(k+1-\beta)w(k+1-\beta) = 0, \qquad m \neq n.$$

According to [12, Theorem 3.3] the recurrence coefficients $\tilde{a}_n^2$ and $\tilde{b}_n$ satisfy system (5), (6) (with tilde) but with initial conditions

$$\tilde{a}_0^2 = 0, \qquad \tilde{b}_0 = \frac{m_1 + \tau\hat{m}_1}{m_0 + \tau\hat{m}_0}, \tag{10}$$

where

$$m_0 = M(\gamma, \beta, c), \qquad m_1 = \frac{\gamma c}{\beta}M(\gamma+1, \beta+1, c),$$
$$\hat{m}_0 = \frac{\Gamma(\beta)\Gamma(\gamma-\beta+1)}{\Gamma(\gamma)\Gamma(2-\beta)}c^{1-\beta}M(\gamma-\beta+1, 2-\beta, c),$$
$$\hat{m}_1 = \frac{\Gamma(\beta)\Gamma(\gamma-\beta+1)}{\Gamma(\gamma)\Gamma(1-\beta)}c^{1-\beta}M(\gamma-\beta+1, 1-\beta, c).$$

Thus, it is shown in [12] that the orthogonal polynomials for the generalized Meixner weight on the lattice $\mathbb{N}$, on the shifted lattice $\mathbb{N}+1-\beta$ and on the bi-lattice $\mathbb{N}\cup(\mathbb{N}+1-\beta)$ have recurrence coefficients a_n^2 and b_n which satisfy the same nonlinear system of discrete (recurrence) equations but the initial conditions are different in each case.

1.2 The fifth Painlevé equation and its Bäcklund transformation

The Painlevé equations possess the so-called Painlevé property: the only movable singularities of the solutions are poles [9]. They are often referred to as nonlinear special functions and have numerous applications in mathematics and mathematical physics.

The fifth Painlevé equation P_V is given by

$$y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{y'}{t} + \frac{(y-1)^2}{t^2}\left(Ay + \frac{B}{y}\right) + \frac{Cy}{t} + \frac{Dy(y+1)}{y-1}, \tag{11}$$

where $y = y(t)$ and A, B, C, D are arbitrary complex parameters. By using a transformation $y(t) \to y(k_1 t)$ we can take the value of the parameter D equal to any non-zero number. There exists a Bäcklund transformation between solutions of the fifth Painlevé equation with $D \neq 0$.

Theorem 2 ([9, Theorem 39.1])**.** *If $y = y(t)$ is the solution the fifth Painlevé equation* (11) *with parameters A, B, C, D, then the transformation*

$$T_{\varepsilon_1,\varepsilon_2,\varepsilon_3} : \ y \to y_1$$

gives another solution $y_1 = y_1(t)$ with new values of the parameters A_1, B_1, C_1, D_1, where

$$\begin{aligned}
&y_1 = 1 - \frac{2dty}{ty' - ay^2 + (a - b + dt)y + b},\\
&A_1 = -\frac{1}{16D}(C + d(1-a-b))^2, \qquad B_1 = \frac{1}{16D}(C - d(1-a-b))^2,\\
&C_1 = d(b-a), \qquad D_1 = D,\\
&a = \varepsilon_1\sqrt{2A}, \qquad b = \varepsilon_2\sqrt{-2B}, \qquad d = \varepsilon_3\sqrt{-2D}, \qquad \varepsilon_j^2 = 1, \qquad j \in \{1,2,3\}.
\end{aligned}$$

See also [11] for a further description of the Bäcklund transformations and the isomorphism of the group of Bäcklund transformations and the affine Weyl group of $A_3^{(1)}$ type.

2 Main results

In this paper we show how to obtain a relation between the recurrence coefficients and the (classical) solutions of the fifth Painlevé equation. The calculations are similar to calculations in [2] but are more involved. The study of initial conditions of the recurrence coefficients for different lattices is also presented. We can summarize the known results and our recent findings concerning the (generalized) Meixner weights as follows.

The weight $(\beta)_k c^k/(k!)$, $\beta > 0$, $0 < c < 1$, is the classical Meixner weight and the recurrence coefficients are known explicitly. The weight $(\beta)_k c^k/(k!)^2$, $\beta > 0$, $c > 0$, is studied in [2] and the recurrence coefficients are related to classical solutions of P_V with parameters $((\beta-1)^2/2, -(\beta+n)^2/2, 2n, -2)$. It is shown in this paper that the recurrence coefficients for the weight $(\gamma)_k c^k/(k!(\beta)_k)$, $c, \beta, \gamma > 0$, are related to the classical solutions of P_V with parameters $((\gamma-1)^2/2, -(\gamma-\beta+n)^2/2, k_1(\beta+n), -k_1^2/2))$, $k_1 \neq 0$.

2.1 Relation to the fifth Painlevé equation and its Bäcklund transformation

First we obtain a nonlinear discrete equation for $v_n(c)$. From equation (5) with n and equation (6) with $n+1$ we eliminate u_{n+1} by computing the resultant. Next, from the obtained equation and (6) with n we eliminate u_n. As a result, we obtain a nonlinear discrete equation for $v_n(c)$ which we denote by

$$F(v_{n-1}, v_n, v_{n+1}, c) = 0. \tag{12}$$

The equation was obtained by using `Mathematica`[1] but it is too long and too complicated too include here explicitly (all enquiries concerning computations can be sent to the first author). We shall show later on that equation (12) can in fact be obtained from the Bäcklund transformation of the fifth Painlevé equation.

Next we derive the differential equation for v_n. In [2] we have used the Toda system. Since the weight w in (3) on the shifted lattice $\mathbb{N}+1-\beta=\{1-\beta,2-\beta,3-\beta,\ldots\}$ is, up to a constant factor, equal to the weight on the original lattice $\mathbb{N}$ with different parameters [12], it can be shown [2, 5] that the recurrence coefficients a_n and b_n as functions of the parameter c satisfy the Toda system

$$\begin{aligned} \left(a_n^2\right)' &:= \frac{d}{dc}\left(a_n^2\right) = \frac{a_n^2}{c}(b_n - b_{n-1}), \\ b_n' &:= \frac{d}{dc} b_n = \frac{1}{c}\left(a_{n+1}^2 - a_n^2\right). \end{aligned} \tag{13}$$

The same system (13) holds for the initial lattice $\mathbb{N}$ [2].

Solving (5) for u_{n+1} and (6) for v_{n-1} and substituting into the Toda system (13) (where we have replaced a_n^2 and b_n by their expressions in terms of u_n and v_n from Theorem 1), we get two equations

$$u_n' = R_1(u_n, v_n, c)$$

and

$$v_n' = R_2(u_n, v_n, c), \tag{14}$$

where the differentiation is with respect to c. The explicit expressions for R_1 and R_2 are again quite complicated but readily computed in `Mathematica`. By differentiating equation (14) and substituting the expression for u_n' we obtain an equation for v_n'', v_n', v_n, u_n. Finally, eliminating u_n between this equation and (14) gives a nonlinear second order second degree equation for v_n:

$$G(v_n'', v_n', v_n, c) = 0. \tag{15}$$

We have again used `Mathematica` to compute this long expression. Now the main difficulty is in identifying this equation. Since (5), (6) is similar to the discrete system in [2], we can try to reduce equation (15) to the fifth Painlevé equation. First, we scale the independent variable $c \to c/k_1$ and denote $c = k_1 t$, $v(c) = V(t)$. The equation (15) becomes $G_1(V_n'', V_n', V_n, t) = 0$, where the differentiation is with respect to t. By considering the Ansatz

$$V_n(t) = \frac{p_1(t)y' + p_2(t)y^2 + p_3(t)y + p_4(t)}{y(y-1)},$$

where $y = y(t)$ is the solution of the fifth Painlevé equation and $p_j(t)$ are unknown functions, we finally get the following theorem.

Theorem 3. *The equation $G_1(V_n'', V_n', V_n, t) = 0$ is reduced to the fifth Painlevé equation* $\mathrm{P_V}$ *by the following transformations:*

1. $$V(t) = \frac{k_1 t(ty' - (1+\beta-2\gamma)y^2 + (1+n-k_1t+\beta-2\gamma)y - n)}{2(\gamma-1)(y-1)y}, \tag{16}$$

   where $y = y(t)$ satisfies $\mathrm{P_V}$ *with*

   $$A = \frac{(\beta-1)^2}{2}, \qquad B = -\frac{n^2}{2}, \qquad C = k_1(n-\beta+2\gamma), \qquad D = -\frac{k_1^2}{2}; \tag{17}$$

[1] `http://www.wolfram.com`

2.

$$V(t) = \frac{k_1 t(ty' - (\beta-\gamma)y^2 + (n-1-k_1t+\beta)y + 1 - n - \gamma)}{2(\gamma-1)(y-1)y}, \tag{18}$$

where $y = y(t)$ *satisfies* $\mathrm{P_V}$ *with*

$$A = \frac{(\beta-\gamma)^2}{2}, \qquad B = -\frac{(\gamma+n-1)^2}{2}, \qquad C = k_1(2+n-\beta), \qquad D = -\frac{k_1^2}{2}; \tag{19}$$

3.

$$V(t) = \frac{k_1 t(ty' + (\gamma-1)y^2 + (1+n-k_1t-\beta)y - n + \beta - \gamma)}{2(\gamma-1)(y-1)y},$$

where $y = y(t)$ *satisfies* $\mathrm{P_V}$ *with*

$$A = \frac{(\gamma-1)^2}{2}, \qquad B = -\frac{(\gamma-\beta+n)^2}{2}, \qquad C = k_1(\beta+n), \qquad D = -\frac{k_1^2}{2}. \tag{20}$$

Remark 1. The parameters (17) are invariant under $\beta \to 2-\beta$, $\gamma \to \gamma+1-\beta$; compare with the parameters in the weight (8).

Remark 2. Cases 2 and 3 in Theorem 3 follow from case 1 by considering the Bäcklund transformation of Theorem 2. Indeed, the compositions of the transformations

$$T_{1,1,-1} \circ T_{1,-1,1} = T_{1,1,1} \circ T_{1,-1,-1}$$

give the transformation

$$Y_1 = y - \frac{2(\gamma-1)(y-1)^2 y}{ty' - (1+\beta-2\gamma)y^2 + (1+n-k_1t+\beta-2\gamma)y - n},$$

where $y = y(t)$ solves $\mathrm{P_V}$ with parameters (17) and Y_1 solves $\mathrm{P_V}$ with (19). Similarly, the compositions of the transformations

$$T_{1,1,-1} \circ T_{-1,-1,1} = T_{1,1,1} \circ T_{-1,-1,-1}$$

give

$$Y_2 = w + \frac{2(\beta-\gamma)(y-1)^2 y}{ty' - (1+\beta-2\gamma)y^2 + (1+n-k_1t+\beta-2\gamma)y - n},$$

where $y = y(t)$ solves $\mathrm{P_V}$ with (17) and Y_2 solves $\mathrm{P_V}$ with (20).

Next we show that equation (12) can, in fact, be obtained from the Bäcklund transformation of $\mathrm{P_V}$ in Theorem 2. Let us take, for instance, the parameters (17) and $k_1 = 1$ for simplicity (hence, $c = t$). Suppose $y = y_n(t)$ solves $\mathrm{P_V}$ with (17). Then by considering the transformations $T_{1,-1,-1} \circ T_{-1,-1,1} \circ T_{-1,-1,1}$ and $T_{1,-1,-1} \circ T_{1,1,-1} \circ T_{1,1,1}$ we get new solutions of $\mathrm{P_V}$ with parameters (17) for $n+1$ and $n-1$ respectively. In particular,

$$\begin{aligned} y_{n+1} = 1 &- \frac{2t(n+\gamma)y}{(\beta-1)(ty' + y(1+n+t-\beta+(\beta-1)y) - n)} \\ &+ \frac{2t(1+n-\beta+\gamma)y}{(\beta-1)(ty' + y(n-1+t+\beta+(1-\beta)y) - n)} \end{aligned}$$

and

$$\begin{aligned} y_{n-1} = 1 &+ \frac{2t(\gamma+n-1)y}{(\beta-1)(ty' - y(1+n+t-\beta+(\beta-1)y) + n)} \\ &- \frac{2t(n-\beta+\gamma)y}{(\beta-1)(ty' - y(n-1+t+\beta+(1-\beta)y) + n)}. \end{aligned}$$

Expressing $v_{n\pm1}$ in terms of y_n by using (16), we can compute that (12) is identically zero. Hence, the discrete system (5), (6) can be obtained from the Bäcklund transformation of $\mathrm{P_V}$.

2.2 Initial conditions

In this section we study the initial conditions (7), (9), (10) for the generalized Meixner weight (3) on the lattice $\mathbb{N}$, on the shifted lattice and on the bi-lattice respectively.

Let $n = 0$. Since $\gamma \neq 1$, we get that $u_0 = 0$ from $a_0^2 = 0$. We also put $k_1 = 1$ and $c = t$. From (14) we have that $v = v_0(t)$ satisfies the first order nonlinear equation

$$t^2 v' = (\gamma - 1)v^2 + t(2 - t + \beta - 2\gamma)v + (\gamma - \beta)t^2. \tag{21}$$

Since $b_0 = \gamma - \beta + t - (\gamma - 1)v_0/t$, we can find v_0 for (7), (9) and (10). We can verify that all of them satisfy (21) using formulas for the confluent hypergeometric functions from [1]. Note that (10) depends on an arbitrary parameter τ.

The fifth Painlevé equation (11) with parameters (19) with $k_1 = 1$ and $n = 0$ has particular solutions which solve the following first order nonlinear equation:

$$ty' = (\beta - \gamma)y^2 + (t - 1 - \beta + 2\gamma)y + 1 - \gamma. \tag{22}$$

Substituting expression (18) in (21) and assuming that y satisfies (22), we indeed find that the equation is satisfied. We also find that $v(t) = t/y(t)$. Thus, the initial conditions (7), (9), (10) for the generalized Meixner weight (3) on the lattice $\mathbb{N}$, on the shifted lattice and on the bi-lattice respectively are related to solutions of the first order differential equation (22), which, in turn, satisfies $\mathrm{P_V}$.

3 A remark on the Bäcklund transformation

In this section we study the Bäcklund transformation of the fifth Painlevé equation ($\mathrm{P_V}$) and find when a linear combination of two solutions is also a solution of $\mathrm{P_V}$. In particular, we show that if y_1 and y_2 are solutions of $\mathrm{P_V}$ obtained from a solution y by certain Bäcklund transformations, then there is a constant $M \neq 0, 1$ such that $My_1 + (1 - M)y_2$ is also a solution of $\mathrm{P_V}$.

Example 1. In [2] it is shown that if $y := y_n(t)$ (related to the recurrence coefficients of the generalized Meixner polynomials) is the solution of (11) with

$$A = \frac{(\beta - 1)^2}{2}, \qquad B = -\frac{(\beta + n)^2}{2}, \qquad C = 2n, \qquad D = -2,$$

then one can show that $y_{n+1} = y_{n+1}(t)$ given by

$$\begin{aligned} y_{n+1} = 1 &+ \frac{4(n+1)ty}{(\beta - 1)(ty' + (2t + 2\beta - 1 + n - (\beta - 1)y)y - n - \beta)} \\ &- \frac{4t(n + \beta)y}{(\beta - 1)(ty' + (1 + n + 2t + (\beta - 1)y)y - n - \beta)} \end{aligned} \tag{23}$$

is the solution of (11) with

$$A = \frac{(\beta - 1)^2}{2}, \qquad B = -\frac{(\beta + n + 1)^2}{2}, \qquad C = 2(n + 1), \qquad D = -2.$$

It can be observed that the transformation (23) can be written in the following form:

$$y_{n+1} = My_1 + (1 - M)y_2, \qquad M = \frac{n + 1}{1 - \beta},$$

where

$$y_1 = T_{1,-1,1}y = 1 + \frac{4ty}{n+\beta-(n-1+2\beta+2t)y+(\beta-1)y^2-ty'}$$

is a solution of (11) with

$$A_1 = \frac{(n+1)^2}{2}, \qquad B_1 = -\frac{1}{2}, \qquad C_1 = -2(2\beta+n-1), \qquad D_1 = -2$$

and

$$y_2 = T_{-1,-1,1}y = 1 + \frac{4ty}{n+\beta-(1+n+2t)y-(\beta-1)y^2-ty'}$$

is a solution of (11) with

$$A_2 = \frac{(n+\beta)^2}{2}, \qquad B_2 = -\frac{\beta^2}{2}, \qquad C_2 = -2(n+1), \qquad D_2 = -2.$$

Similarly, if

$$y_1 = T_{1,1,-1}y = 1 + \frac{4ty}{n+\beta-(1+n+2t)y-(\beta-1)y^2+ty'},$$
$$y_2 = T_{-1,1,-1}y = 1 + \frac{4ty}{n+\beta-(n-1+2\beta+2t)y+(\beta-1)y^2+ty'}$$

are solutions of (11) with

$$A_1 = \frac{(\beta+n-1)^2}{2}, \qquad B_1 = -\frac{(\beta-1)^2}{2}, \qquad C_1 = -2(n+1), \qquad D_1 = -2$$

and

$$A_2 = \frac{n^2}{2}, \qquad B_2 = 0, \qquad C_2 = -2(2\beta+n-1), \qquad D_2 = -2,$$

respectively, then

$$y_{n-1} = My_1 + (1-M)y_2, \qquad M = \frac{(\beta+n-1)}{\beta-1}$$

is a solution of (11) with

$$A = \frac{(\beta-1)^2}{2}, \qquad B = -\frac{(\beta+n-1)^2}{2}, \qquad C = 2(n-1), \qquad D = -2.$$

Such observations motivate us to study the question when the sum $My_1 + (1-M)y_2$ of two solutions of $\mathrm{P_V}$ is also a solution of the same equation. Clearly, we impose the conditions that $M \neq 0$ and $M \neq 1$.

Theorem 4. *Let $y = y(t)$ be a solution of P_{V} with parameters A, B, C, $D = -2$ and*

$$y_1 = T_{\varepsilon_1,\varepsilon_2,\varepsilon_3}y, \qquad y_2 = T_{\delta_1,\delta_2,\delta_3}y,$$

where $\varepsilon_j^2 = \delta_j^2 = 1$ and $\varepsilon_j \neq \delta_j$ for some $j \in \{1,2,3\}$. Then

$$v = M\,y_1 + (1-M)\,y_2$$

with $M \neq 0; 1$ is a solution of $\mathrm{P_V}$ *with parameters A_v, B_v, C_v, $D_v = -2$ in the following cases:*

1. $\delta_1 = \varepsilon_1$, $\delta_2 = -\varepsilon_2$, $\delta_3 = \varepsilon_3$ *and*

$$M = \frac{2\varepsilon_1\sqrt{2A} + 2\varepsilon_2\sqrt{-2B} - \varepsilon_3 C - 2}{4\varepsilon_2\sqrt{-2B}},$$
$$A_v = -B, \qquad B_v = \frac{2\varepsilon_1\sqrt{2A} - 2A - 1}{2}, \qquad C_v = -C - 2\varepsilon_3;$$

2. $\delta_1 = -\varepsilon_1$, $\delta_2 = \varepsilon_2$, $\delta_3 = \varepsilon_3$ *and*

$$M = \frac{2\varepsilon_1\sqrt{2A} + 2\varepsilon_2\sqrt{-2B} - \varepsilon_3 C - 2}{4\varepsilon_1\sqrt{2A}},$$
$$A_v = A, \qquad B_v = \frac{2\varepsilon_2\sqrt{-2B} + 2B - 1}{2}, \qquad C_v = C + 2\varepsilon_3.$$

Proof. The proof of this result is computational. We first obtain that in case $\delta_3 = -\varepsilon_3$ one gets only cases $M = 0$ and $M = 1$. In case $\delta_3 = \varepsilon_3$ one needs to consider 3 cases separately (since $\delta_1 = \varepsilon_1$, $\delta_2 = \varepsilon_2$ gives a trivial result for the function v): $\delta_1 = \varepsilon_1$, $\delta_2 = -\varepsilon_2$; $\delta_1 = -\varepsilon_1$, $\delta_2 = \varepsilon_2$; $\delta_1 = -\varepsilon_1$, $\delta_2 = -\varepsilon_2$. However, in the last case we get $M = 0$ or $M = 1$. ■

The examples at the beginning of the section correspond to the second case of the theorem. Similarly, we can get expressions for y_{n+1} and y_{n-1} in the previous section by using this theorem.

Various properties of the repeated application of the Bäcklund transformations are studied in [6, 7, 8]. Repeated applications of the Bäcklund transformations to the seed solutions usually lead to very cumbersome formulas. However, as shown in this section, we can get linear dependence between three solutions. Moreover, our formulas suggest that the function v has the same poles as y_1 and y_2 and, thus, they can be useful to study various properties of the solutions. Other Painlevé equations might have similar properties so one can try to study when for instance a linear combination of several solutions or a product or a cross-ratio of several solutions is also a solution (see also the representation of solutions in [11]). Although computationally difficult, this deserves further study.

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